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Algebra related to elementary particles of spin $\frac{3}{2}$

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The algebra generated by the four matrices β_μ occurring in the relativistic wave equation of a particle of maximum spin n on the basis of the commutation rules for these matrices obtained previously by one of the authors has been investigated. Auxiliary quantities η_μ satisfying the equations (5) are introduced. These η_μ are given as polynomials in β_μ. With the help of these, further auxiliary quantities $\xi_\mu = \eta_\mu \beta_\mu$ are defined. It is shown that for half odd integral spin, the ξ's and η's form two mutually commuting sets of symbols of which the η's satisfy the same commutation rules as the Dirac matrices. This proves that the algebra in the case of half odd integral spin is the direct product of the Dirac algebra and an associated ξ-algebra. For the special case of maximum spin $\frac{3}{2}$ the ξ-algebra has been studied in detail, and it is shown that this algebra has just three representations of orders 1, 4, 5 such that $1^2+4^2+5^2 = 42$ = rank of the algebra. Explicit representations are given in the non-trivial cases of orders 4 and 5. The 4-row representation of the ξ's gives a representation of the β's of order 16 which is likely to be of importance in connexion with Bhabha's new theory of the proton.

1. Introduction

One of us (B.S.M.) has considered (1942*a*, referred to here as I) the question of deriving commutation rules for the matrices β_μ appearing in the relativistic wave equation of elementary particles of arbitrary spin in the form‡

$$\partial_\mu \beta_\mu \psi + \chi\psi = 0, \tag{1}$$

and shown that this problem can be solved in the general case by postulating that the spin operator $t_{\mu\nu} = is_{\mu\nu}$ satisfies the equation (I (9*a*))

$$t_{\mu\nu} = (\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \equiv (\beta_\mu, \beta_\nu). \tag{2}$$

The general commutation rules, valid for all spins, can then be written (I (11*a*)) as

$$(\beta_\mu, t_{\nu\rho}) = (\beta_\mu, (\beta_\nu, \beta_\rho)) = \delta_{\mu\nu}\beta_\rho - \delta_{\mu\rho}\beta_\nu. \tag{3}$$

The special cases of spins $\frac{3}{2}$ and 2 were considered in I and the restricted forms of (3), on the further assumption that the eigen-values of $s_{\mu\nu}$ for a particle of spin n are $n, n-1, n-2, \ldots, -n+1, -n$, were also given there for these cases (I (26) and (34)).

The imposition of condition (2), while it solves the problem of deriving commutation rules, has also very far-reaching consequences in that it makes the wave equation (1) itself of fundamental importance in obtaining properties of elementary particles. Equations of the form (1) for which condition (2) is satisfied can be considered as natural generalizations, to general spin values, of the corresponding equations for the well-known cases of the electron and the meson. Bhabha has recently considered§ (1945*a, b, c*) the implications of the assumption (2), and set

‡ The notation adopted in this paper is the same as that of Kemmer (1939) with χ denoting the rest mass instead of his κ.

§ We wish to thank Professor Bhabha for allowing us to see a manuscript of his paper (1945*b*) before publication.

up a general theory of relativistic wave equations of the type (1) wherein he has shown that the problem of finding all irreducible equations of the form (1) can be connected with that of finding all irreducible representations of the Lorentz group in five dimensions. He has further shown that on this theory, a particle of spin n must appear with n different values of the rest mass‡ if n is an integer, and $n+(\frac{1}{2})$ values if n is half an odd integer, the higher values being simple rational multiples of the lowest value.

On the basis of his theory Bhabha has considered (1945*c*, p. 261) in particular, the two possible equations of the form (1) for a particle of maximum spin $\frac{3}{2}$, and indicated that the equation given by one of the representations of the Lorentz group in five dimensions denoted by $R_5(\frac{3}{2}, \frac{1}{2})$ may possibly describe the behaviour of the proton. The degree of this representation, viz. 16, and an explicit expression for the same have also been derived by him.

We have independently investigated the algebra generated by the symbols β_μ and 1 (unit element) governed by the commutation rules for the β_μ, and shown that, in the case of half odd integral spin, this algebra is the direct product of the Dirac algebra, and another associated algebra which we call a ξ-algebra. In particular, for the case of spin $\frac{3}{2}$ the corresponding ξ-algebra is of rank 42, and the Dirac algebra being of rank 16, the original algebra is of rank $16\times 42 = 672$. The circumstance of this algebra being a direct product simplifies the work to a great extent, and makes it unnecessary to enumerate the 672 linearly independent elements generated by the β_μ and 1. It is shown that the ξ-algebra has three irreducible representations of orders 1, 4, 5 respectively, $(42 = 1^2+4^2+5^2)$, so that for the original algebra of spin $\frac{3}{2}$ there are also three irreducible representations of orders (4×1), (4×4), (4×5), i.e. 4, 16, 20. Of these the first refers to the well-known representation of the α-matrices of Dirac.

The commutation rules for spin $\frac{3}{2}$ are likely to prove useful in investigations relating to the above theory of Bhabha for the proton, and the associated ξ-algebra which is easier to handle than the original β-algebra might be used in calculations. We have obtained representation matrices for the ξ-algebra both of orders 4 and 5, such that the matrices are real and symmetric and thus Hermitian. A table of spurs of the elements of the basis of the ξ-algebra in both the representations is also appended.

2. The auxiliary quantities η_μ, ξ_μ-algebra for half odd integral spin as a direct product

The commutation rules satisfied by the β_μ are given by (3), and the further assumption that the eigen-values of the spin operator $s_{\mu\nu}$ for a particle of spin n are $n, n-1, \ldots, -n+1, -n$, or that $s_{\mu\nu}$ satisfies the characteristic equation

$$(s_{\mu\nu}^2 - n^2)(s_{\mu\nu}^2 - \overline{n-1}^2)\ldots = 0$$

‡ See also Madhavarao (1942*b*, 1945).

leads, in virtue of (2), to the result that β_μ also satisfies the same characteristic equation

$$(\beta_\mu^2 - n^2)(\beta_\mu^2 - \overline{n-1}^2)\ldots = 0, \tag{4}$$

the last factor being β_μ or $(\beta_\mu^2 - 1/4)$ according as n is an integer or half an odd integer. This is an immediate consequence of the relations

$$(\beta_\mu, \beta_\nu) = t_{\mu\nu}; \quad (\beta_\mu, t_{\mu\nu}) = \beta_\nu; \quad (t_{\mu\nu}, \beta_\nu) = \beta_\mu$$

which follow from (2) and (3) (Bhabha 1945*a*; Madhavarao 1942*b*, 1945).

We now proceed to investigate the algebra generated by the symbols β_μ satisfying the commutation rules (3) and (4). We first of all establish the existence of a symbol η_μ satisfying the relations

$$\left.\begin{aligned} \eta_\mu^2 &= 1, \\ \eta_\mu\beta_\nu + \beta_\nu\eta_\mu &= 0 \quad (\mu \neq \nu), \\ \eta_\mu\beta_\mu &= \beta_\mu\eta_\mu \quad \text{(no summation)}, \end{aligned}\right\} \tag{5}$$

Before proceeding to prove this, we might remark that the introduction of η_μ is suggested by considering the possibility of passing from the wave equation (1) to the adjoint wave equation

$$\partial_\mu \psi^\dagger \beta_\mu - \chi\psi^\dagger = 0 \tag{1†}$$

with the definition $\psi^\dagger = i\psi^*\eta_4$. This is possible if we assume, as is compatible with (3) and (4), that the β_μ are Hermitian‡ (in analogy with the Dirac matrices), if $\eta_4^2 = 1$, $\eta_4\beta_4 = \beta_4\eta_4$, and $\eta_4\beta_k + \beta_k\eta_4 = 0$ $(k = 1, 2, 3)$ which suggest the relations (5).

As indicated by (4), the minimal equation for β_μ is a polynomial of degree $2n+1$, and we take $\eta_\mu = f(\beta_\mu)$ as a polynomial of degree $2n$ in β_μ, an assumption which is compatible with the relations (5). These relations further enable us to determine the coefficients of this polynomial.

Taking β_μ diagonal in an irreducible representation, the commutation rule (3) in the form

$$(\beta_\mu, (\beta_\mu, \beta_\nu)) = \beta_\nu \quad (\mu \neq \nu),$$

leads to the result that the matrix for β_ν has all its elements equal to zero except those in the diagonals above and below the principal diagonal. With this form for β_ν, the relation $\eta_\mu\beta_\nu + \beta_\nu\eta_\mu = 0$ shows, since the representation is irreducible, that

$$f(n) = -f(n-1) = f(n-2) = -f(n-3) = \ldots.$$

Also $\eta_\mu^2 = 1$ gives, since β_μ is taken diagonal and $\eta_\mu = f(\beta_\mu)$,

$$\{f(n)\}^2 = \{f(n-1)\}^2 = \ldots = 1.$$

‡ That this is so has been proved by Bhabha on general considerations (1945*c*, p. 245).

Thus $f(n)$ satisfies the conditions

$$f(n) = -f(n-1) = f(n-2) = \ldots = 1 \quad (\text{or } -1),$$

i.e.

$$f(n-r) = (-1)^r \quad (r = 0, 1, 2, \ldots, 2n).$$

Since $f(x)$ is of degree $2n$ in x, it is uniquely determined by the above condition and can be written down at once, by using Lagrange's interpolation formula, in the form

$$f(x) = \frac{(x-n)(x-n+1)\ldots(x+n-1)(x+n)}{(2n)!} \sum_{r=0}^{2n} \binom{2n}{r} \frac{1}{(x-n+r)}. \tag{6}$$

For the case of half odd integral spin $n-\frac{1}{2}$, this gives

$$f(x) = \frac{2x(x^2-1/4)(x^2-9/4)\ldots(x^2-\overline{2n-1}^2/4)}{(2n-1)!} \sum_{r=1}^{n} \binom{2n-1}{n-r} \frac{1}{(x^2-\overline{2r-1}^2/4)}, \tag{6a}$$

while for integral spin n

$$f(x) = \frac{(x^2-1^2)(x^2-2^2)\ldots(x^2-n^2)}{(n!)^2} + \frac{2x^2(x^2-1^2)\ldots(x^2-n^2)}{(2n)!} \sum_{r=1}^{n} \binom{2n}{n-r} \frac{1}{(x^2-r^2)}. \tag{6b}$$

Equations (6a) and (6b) show that η_μ contains only odd powers of β_μ in the half odd integral, and only even powers in the integral case. This fact, together with $\eta_\mu \beta_\nu + \beta_\nu \eta_\mu = 0$, shows that

$$\left.\begin{aligned} \eta_\mu \eta_\nu + \eta_\nu \eta_\mu &= 0 \quad \text{for half odd integral spin,} \\ \eta_\mu \eta_\nu - \eta_\nu \eta_\mu &= 0 \quad \text{for integral spin,} \end{aligned}\right\} \tag{7}$$

i.e. η_μ and η_ν *commute or anti-commute according as the spin is integral or half odd integral.*

The η_4 introduced here is the same as the metric operator D of Bhabha (1945c (14)) given by $D = e^{i\pi\alpha^0}$ as the representative in five dimensions of the transformation of the Lorentz group reversing the axes 1, 2, 3 and interpreted as a rotation in the 04 plane through the angle π. Here similarly we can interpret η_μ as the rotation in the $\mu 5$ plane through an angle π.

We next introduce a second auxiliary symbol ξ_μ defined by

$$\xi_\mu = \eta_\mu \beta_\mu = \beta_\mu \eta_\mu \quad (\text{no summation}). \tag{8}$$

Using $\eta_\mu^2 = 1$, this leads immediately to

$$\xi_\mu \eta_\mu = \eta_\mu \xi_\mu = \beta_\mu. \tag{9}$$

Finally from $\eta_\mu \beta_\nu + \beta_\nu \eta_\mu = 0$ $(\mu \neq \nu)$ and (7), we have

$$\left.\begin{aligned} \xi_\mu \eta_\nu - \eta_\nu \xi_\mu &= 0 \quad \text{for half odd integral spin,} \\ \xi_\mu \eta_\nu + \eta_\nu \xi_\mu &= 0 \quad \text{for integral spin,} \end{aligned}\right\} \tag{10}$$

i.e. ξ_μ and η_ν *commute or anti-commute according as the spin is half odd integral or integral.*

We tabulate below the several algebraic relations between the β_μ, η_μ and ξ_μ:

$$\left.\begin{array}{c}
\eta_\mu^2 = 1 \\
\eta_\mu \beta_\nu + \beta_\nu \eta_\mu = 0 \\
\eta_\mu \beta_\mu = \beta_\mu \eta_\mu = \xi_\mu \\
\eta_\mu \xi_\mu = \xi_\mu \eta_\mu = \beta_\mu \\
\xi_\mu^2 = \beta_\mu^2 \\
\begin{array}{c|c}
\eta_\mu \eta_\nu + \eta_\nu \eta_\mu = 0 & \eta_\mu \eta_\nu - \eta_\nu \eta_\mu = 0 \\
\eta_\mu \xi_\nu - \xi_\nu \eta_\mu = 0 & \eta_\mu \xi_\nu + \xi_\nu \eta_\mu = 0 \\
\text{for half odd integral spin} & \text{for integral spin}
\end{array}
\end{array}\right\} \tag{11}$$

The proof that in the case of half odd integral spin the algebra is a direct product is now immediate. Consider any expression which is a product of the powers of the several β_μ. Substituting $\beta_\mu = \eta_\mu \xi_\mu = \xi_\mu \eta_\mu$, this expression separates out, in virtue of

$$\xi_\mu \eta_\mu = \eta_\mu \xi_\mu, \quad \xi_\mu \eta_\nu - \eta_\nu \xi_\mu = 0,$$

into the product of two parts, one expressed in terms of the ξ's alone, and the other in terms of the η's alone. Further, the same substitution reduces the commutation rules (3) and (4) into rules involving the ξ's only, the η's falling out by using the relations $\eta_\mu^2 = 1$ and $\eta_\mu \eta_\nu + \eta_\nu \eta_\mu = 0$. These last relations however show that the η_μ's with their powers and products and the unit element 1 form a sub-algebra identical with the Dirac algebra, say D_η and another generated by the ξ's which might be called a ξ-algebra and denoted by A_ξ. Thus

$$A = D_\eta \times A_\xi. \tag{12}$$

3. The ξ-algebra for the case of spin $\frac{3}{2}$

For the particular case of spin $\frac{3}{2}$, the characteristic equation (4) reduces to

$$\beta_\mu^4 - \tfrac{5}{2}\beta_\mu^2 + \tfrac{9}{16} = 0, \tag{13}$$

and this and (3) constitute the set of commutation rules for the β_μ. These can be written in such a way that they contain terms consisting of products of four β's, on the one hand, and connect these with terms containing products of two and zero β's on the other. Such types of rules can be derived from (13) by taking Poisson brackets with the $t_{\mu\nu}$ successively, and using (2) and (3). There are five such types of rules depending on the number of equal and unequal indices appearing in them. All the types including (13) can be derived from a general commutation rule: (I (26))

$$\begin{aligned}
2(\beta_\mu\beta_\nu\beta_\rho\beta_\epsilon + \beta_\mu\beta_\epsilon\beta_\rho\beta_\nu + \beta_\nu\beta_\rho\beta_\epsilon\beta_\mu + \beta_\epsilon\beta_\rho\beta_\nu\beta_\mu) \\
= 3(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\,\delta_{\rho\epsilon} + 3(\beta_\mu\beta_\epsilon + \beta_\epsilon\beta_\mu)\,\delta_{\nu\rho} \\
+ (\beta_\rho\beta_\epsilon + \beta_\epsilon\beta_\rho)\,\delta_{\mu\nu} + (\beta_\nu\beta_\rho + \beta_\rho\beta_\nu)\,\delta_{\mu\epsilon} \\
+ (\beta_\nu\beta_\epsilon + \beta_\epsilon\beta_\nu)\,\delta_{\mu\rho} + (\beta_\mu\beta_\rho + \beta_\rho\beta_\mu)\,\delta_{\nu\epsilon} \\
- \tfrac{3}{2}(\delta_{\mu\nu}\delta_{\rho\epsilon} + \delta_{\mu\rho}\delta_{\nu\epsilon} + \delta_{\nu\rho}\delta_{\mu\epsilon}).
\end{aligned} \tag{14}$$

The individual types of rules are given in I, but for our present purposes, we write only two of them, corresponding to $\mu \neq \nu = \rho = \epsilon$, and μ, ν, ρ, ϵ unequal, i.e.

$$4(\beta_\mu^3\beta_\nu+\beta_\nu\beta_\mu^3) = 7(\beta_\mu\beta_\nu+\beta_\nu\beta_\mu) \tag{15}$$

and
$$\beta_\mu\beta_\nu\beta_\rho\beta_\epsilon+\beta_\mu\beta_\epsilon\beta_\rho\beta_\nu+\beta_\nu\beta_\rho\beta_\epsilon\beta_\mu+\beta_\epsilon\beta_\rho\beta_\nu\beta_\mu = 0. \tag{16}$$

The commutation rule (15) suggests immediately that the auxiliary symbol η_μ can be taken, in virtue of $\eta_\mu\beta_\nu+\beta_\nu\eta_\mu = 0$, as

$$\eta_\mu = \text{const.}\,(4\beta_\mu^3-7\beta_\mu)$$

and $\eta_\mu^2 = 1$ gives, using (13), this constant equal to $\pm\frac{1}{3}$. We therefore take

$$\eta_\mu = \tfrac{1}{3}(4\beta_\mu^3-7\beta_\mu) \tag{17}$$

which is also obtained from (6*a*) by putting $n = 2$ in that equation.

The auxiliary ξ_μ is given by $\xi_\mu = \eta_\mu\beta_\mu$ and, using (17) and (13), this gives $\xi_\mu = \beta_\mu^2-\frac{3}{4}$ and, since $\xi_\mu^2 = \beta_\mu^2$, we have

$$\xi_\mu^2 = \xi_\mu+\tfrac{3}{4}, \tag{18}$$

which is a very important equation since it materially simplifies the ξ-algebra.

We can now derive the commutation rules for the ξ_μ by substituting $\beta_\mu = \xi_\mu\eta_\mu = \eta_\mu\xi_\mu$ in the commutation rules for the β_μ and removing the η_μ by using the relations $\eta_\mu^2 = 1$, $\eta_\mu\eta_\nu+\eta_\nu\eta_\mu = 0$ and $\eta_\nu\xi_\mu-\xi_\nu\eta_\mu = 0$. This can be done for all the types of rules derivable from (14), but it is however remarkable that by using (18) and the general rule (3) expressed in terms of the ξ's, all these types of commutation rules are identically satisfied with the single exception of (16). The commutation rules for the ξ_μ therefore consist of (18) and (3) and (16) expressed in terms of the ξ's. Using the notation $(a,b)\equiv ab-ba$ and $[a,b]\equiv ab+ba$, (3) expressed in the ξ's gives the two rules

$$[\xi_\mu,[\xi_\mu,\xi_\nu]] = \xi_\nu \;\; (\mu\neq\nu) \quad \text{and} \quad (\xi_\mu,[\xi_\nu,\xi_\rho]) = 0 \;\; (\mu,\nu,\rho\neq), \tag{19a,b}$$

and (16) in terms of the ξ's becomes

$$\xi_\mu(\xi_\nu\xi_\rho\xi_\epsilon-\xi_\epsilon\xi_\rho\xi_\nu) = (\xi_\nu\xi_\rho\xi_\epsilon-\xi_\epsilon\xi_\rho\xi_\nu)\,\xi_\mu \tag{20}$$

with μ, ν, ρ, ϵ unequal.

(18), (19) and (20) constitute the commutation rules for the ξ's. We can write them in further abbreviated form by writing

$$\xi_{\mu\nu\rho}\equiv\xi_\mu\xi_\nu\xi_\rho \quad \text{and} \quad (\xi_{\mu\nu\rho})\equiv(\xi_{\mu\nu\rho}-\xi_{\rho\nu\mu}),$$

and using a similar notation for products containing two or four terms. The rules then become

$$\left.\begin{aligned} \xi_\mu^2 &= \xi_\mu+\tfrac{3}{4}, \\ \xi_{\mu\nu}+\xi_{\nu\mu}+2\xi_{\mu\nu\mu} &= -\tfrac{1}{2}\xi_\mu \quad (\mu\neq\nu), \\ (\xi_{\mu\nu\rho}) = (\xi_{\nu\rho\mu}) &= (\xi_{\rho\mu\nu}) \quad (\mu\neq\nu\neq\rho), \\ \xi_\mu(\xi_{\nu\rho\epsilon}) &= (\xi_{\nu\rho\epsilon})\,\xi_\mu \quad (\mu\neq\nu\neq\rho\neq\epsilon). \end{aligned}\right\} \tag{21a–d}$$

It is now easy to enumerate the number of independent elements of the algebra A_ξ. In virtue of (21a) there are only twelve elements of the type $\xi_{\mu\nu}$. From (21c) we see that there are eight relations among the twenty-four products $\xi_{\mu\nu\rho}$ for which $\mu \neq \nu \neq \rho$, and hence there are only sixteen elements of this type. Also the elements $\xi_{\mu\nu\rho}$ for which two or more indices are equal would be enumerated under ξ_μ or $\xi_{\mu\nu}$ according to (21 a, b). Similar remarks apply to $\xi_{\mu\nu\rho\epsilon}$ when two or more indices are equal. We have therefore to enumerate such terms when all the indices are different. The number of relations among the twenty-four products $\xi_{\mu\nu\rho\epsilon}$ can be enumerated by writing (21d) using (21c) in the form

$$\left.\begin{aligned} \xi_\mu(\xi_{\nu\rho\epsilon}) &= (\xi_{\nu\rho\epsilon})\,\xi_\mu \\ = \xi_\mu(\xi_{\rho\epsilon\nu}) &= (\xi_{\rho\epsilon\nu})\,\xi_\mu \\ = \xi_\mu(\xi_{\epsilon\nu\rho}) &= (\xi_{\epsilon\nu\rho})\,\xi_\mu \end{aligned}\right\} \tag{22}$$

with similar relations having ξ_ν, ξ_ρ, ξ_ϵ in place of the ξ_μ, and giving in all four sets of five relations of the type (22). It is, however, easily verified that of these four sets, any one can be obtained from the remaining three by mere addition, so that there are only fifteen relations among the $\xi_{\mu\nu\rho\epsilon}$ and hence only nine independent elements of that type.

The complete list of independent elements of A_ξ in consonance with (21) is thus given by the table:

$$\left.\begin{array}{lc} \text{element} & \text{number of type} \\ I & 1 \\ \xi_\mu & 4 \\ \xi_{\mu\nu} & 12 \\ \xi_{\mu\nu\rho} & 16 \\ \xi_{\mu\nu\rho\epsilon} & 9 \\ \quad\text{total number} & 42 \end{array}\right\} \tag{23}$$

Hence it follows from (12) that the number of independent elements of the original algebra A is $16 \times 42 = 672$, the Dirac algebra being of rank 16.

Let us find the elements of A_ξ which commute with all the others. We write

$$\left.\begin{aligned} \alpha &= \Sigma\xi_\mu, \\ \beta &= \Sigma\xi_{\mu\nu}, \\ \gamma &= \Sigma\xi_{\mu\nu\rho}, \\ \delta &= \Sigma\xi_{\mu\nu\rho\epsilon}, \end{aligned}\right\} \tag{24}$$

the indices being unequal in each of the summations. By successively postulating commutability with each of the ξ_μ, we find that the following expressions have the desired property:

$$I \text{ (the unit element)}, \quad M = \alpha - \beta, \quad N = 2\gamma - \delta. \tag{25}$$

Any further expressions which might be shown to commute with all the forty-two elements can be proved to be linear combinations of 1, M, N. We also note that the elements α, β, γ, δ form a commutative sub-algebra with the following relations:

$$\left.\begin{array}{l|l} & 2\alpha\beta = 2\beta\alpha = 3\alpha+2\gamma \\ & 4\beta\gamma = 4\gamma\beta = 15\alpha+2\gamma \\ \alpha^2 = \alpha+\beta+3 & 2\gamma\alpha = 2\alpha\gamma = 5\beta+2\gamma+2\delta \\ 2\beta^2 = 2\beta+2\delta+9 & \alpha\delta = \delta\alpha = \gamma \\ 4\gamma^2 = 15\alpha+5\beta-4\gamma-4\delta+45 & 2\beta\delta = 2\delta\beta = 5\beta-4\delta \\ 4\delta^2 = -20\beta+16\delta+45 & 4\gamma\delta = 4\delta\gamma = 15\alpha-4\gamma \end{array}\right\} \tag{26}$$

from which follow the relations

$$\left.\begin{aligned} M^2 &= \tfrac{9}{2}-M-N, \\ MN &= -\tfrac{15}{2}M, \\ N^2 &= \tfrac{225}{4}. \end{aligned}\right\} \tag{27}$$

The primitive idempotent elements of the centre of the ξ-algebra e_1, e_2, e_3 satisfying

$$e_1+e_2+e_3 = 1, \quad e_1e_2 = e_2e_3 = e_1e_3 = 0,$$

are given by

$$\left.\begin{aligned} 30e_1 &= 2N+15, \\ 48e_2 &= 6M-2N+15, \\ -80e_3 &= 10M+2N-15. \end{aligned}\right\} \tag{28}$$

As the algebra A_ξ has the unit element

$$1 = e_1+e_2+e_3,$$

where

$$e_ie_k = \delta_{ik}e_k \quad (i, k = 1, 2, 3),$$

it follows (see van der Waarden 1931, II, p. 161, ex. 4) that A_ξ is the direct sum of the left ideals $A_\xi e_1$, $A_\xi e_2$ and $A_\xi e_3$, and hence from the principal theorem relating to semi-simple rings (ibid. p. 156) we conclude that A_ξ is semi-simple.

Therefore the number of irreducible representations is equal to the rank of the centrum which is *three* in our case. If n_1, n_2, n_3 be the degrees of these representations we have

$$n_1^2+n_2^2+n_3^2 = 42, \tag{29}$$

the algebra A_ξ being of rank 42. The equation (29) has the unique solution in integers given by $n_1 = 4$, $n_2 = 5$, $n_3 = 1$, and thus the algebra has three and only three inequivalent irreducible representations D_1, D_4, D_5 of degrees 1, 4 and 5 respectively.

The left ideals $A_\xi e_1$, $A_\xi e_2$ and $A_\xi e_3$ are of ranks 16, 25 and 1 respectively as could be easily verified. Thus the representations of order 1, 4, 5 are generated respectively by the idempotent elements e_3, e_1, e_2.

4. Representations, and corresponding spurs

We now proceed to give the explicit representations of degrees four and five of A_ξ. The one-dimensional representation in which every ξ_μ is represented by $-\frac{1}{2}$ leads just to the Dirac matrices for the representation of A, and need not be specially considered here.

Fourth order. The eigen-values of ξ_μ are given by (21a) as $-\frac{1}{2}$ and $\frac{3}{2}$, and taking ξ_4 diagonal, we will show that consistent with the commutation rules (21) the form of ξ_4 is uniquely determined by taking the eigen-value $-\frac{1}{2}$ repeated thrice and $\frac{3}{2}$ appearing only once.

Taking ξ_4 diagonal, we can write it in the form

$$\xi_4 = \left(\begin{array}{c|c} -\frac{1}{2}E_{mm} & 0 \\ \hline 0 & \frac{3}{2}E_{nn} \end{array}\right),$$

where E_{mm} denotes the $m \times m$ unit matrix, and $m+n = 4$. Let

$$\xi_1 = \left(\begin{array}{c|c} A_{mm} & B_{mn} \\ \hline C_{nm} & D_{nn} \end{array}\right),$$

where A_{ij} denotes a matrix of i rows and j columns. From the commutation rule (21b) or (19a)

$$\xi_1 = [\xi_4, [\xi_4, \xi_1]] \quad \text{and} \quad \xi_4 = [\xi_1, [\xi_4, \xi_1]]$$

and, using the above matrix forms for ξ_4 and ξ_1, we get the conditions

$$D_{nn} = 0, \tag{29a}$$

$$C_{nm}B_{mn} = \tfrac{3}{4}E_{nn}, \tag{29b}$$

$$A_{mm} = 2B_{mn}C_{nm} - \tfrac{1}{2}E_{mm}, \tag{29c}$$

in deriving which use has also been made of $\xi_1^2 = \xi_1 + \frac{3}{4}$.

From (29b) we obtain that $C_{nm}B_{mn}$ must be of rank n; hence $m \geqslant n$, i.e. in a D_4, $-\frac{1}{2}$ must appear at least twice as an eigen-value of ξ_4. We prove that it must occur thrice, for if it occurs twice, $m = n = 2$, and we have from (29b) and (29c)

$$C_{nn} = \tfrac{3}{4}B_{nn}^{-1} \quad \text{and} \quad A_{nn} = \tfrac{3}{2}B_{nn}B_{nn}^{-1} - \tfrac{1}{2}E_{nn} = E_{nn},$$

i.e. ξ_1 is of the form

$$\xi_1 = \left(\begin{array}{c|c} E_{22} & X \\ \hline \frac{3}{4}X^{-1} & 0 \end{array}\right).$$

Arguing similarly with ξ_2, ξ_3 in place of ξ_1 these would be of the form

$$\xi_2 = \left(\begin{array}{c|c} E_{22} & Y \\ \hline \frac{3}{4}Y^{-1} & 0 \end{array}\right), \quad \xi_3 = \left(\begin{array}{c|c} E_{22} & Z \\ \hline \frac{3}{4}Z^{-1} & 0 \end{array}\right).$$

Using the commutation rule (21c) or (19b), we have $(\xi_4, [\xi_1, \xi_2]) = 0$ which gives $X+Y = 0$. Similarly $(\xi_4, [\xi_2, \xi_3]) = 0$ and $(\xi_4, [\xi_3, \xi_1]) = 0$ give $Y+Z = 0$ and

$Z+X=0$. Thus $X=Y=Z=0$, but this contradicts the fact that the minimal equation of ξ_μ is $\xi_\mu^2-\xi_\mu-\frac{3}{4}=0$. Hence $m=n=2$ is not possible and the only possible case is $m=3$, $n=1$ so that $-\frac{1}{2}$ appears thrice and $\frac{3}{2}$ only once in ξ_4.

We notice that this choice is in consonance with the representation of order 16 given by Bhabha (1945 c, p. 261) and denoted in the five-dimensional notation as $R_5(\frac{3}{2},\frac{1}{2})$. As shown there, the eigen-value of α^0 (corresponding to our β_4) taken in the diagonal form as $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$ which occur 2, 6, 6, 2 times respectively.

With
$$\xi_4=\left(\begin{array}{c|c} -\frac{1}{2}E_{33} & 0 \\ \hline 0 & \frac{3}{2}\end{array}\right)$$

we can now write ξ_k $(k=1,2,3)$, using (29), in the form
$$\xi_k=\left(\begin{array}{c|c} 2B_{31}^{(k)}C_{13}^{(k)}-\frac{1}{2}E_{33} & B_{31}^{(k)} \\ \hline C_{31}^{(k)} & 0\end{array}\right),$$

where $C_{13}^{(k)}B_{31}^{(k)}=\frac{3}{4}$. As we wish to take a real, symmetric (Hermitian) representation we take $C_{13}^{(k)}=(B_{31}^{(k)})'$ (the dash denoting the transpose) and obtain
$$\xi_k=\left(\begin{array}{c|c} 2P_kP'_k-\frac{1}{2} & P_k \\ \hline P'_k & 0\end{array}\right), \tag{30}$$

where P_k is a one-column matrix
$$P_k=\begin{pmatrix} p_k \\ q_k \\ r_k\end{pmatrix}, \tag{31}$$

and P'_k its transpose, satisfying the relations
$$P'_kP_k=\tfrac{3}{4},\quad P'_iP_k=\tfrac{1}{4}\quad (i\neq k),\ (i,k=1,2,3), \tag{32}$$

the latter being a consequence of $(\xi_4,[\xi_i,\xi_k])=0$.

We make a particular choice of the p, q, r, and derive the representation:
$$\left.\begin{array}{ll}
\xi_1=\left(\begin{array}{c|c|c|c} c & \frac{1}{2} & \cdot & c \\ \hline \frac{1}{2} & -s & \cdot & s \\ \hline \cdot & \cdot & -\frac{1}{2} & \cdot \\ \hline c & s & \cdot & \cdot \end{array}\right) &
\xi_2=\left(\begin{array}{c|c|c|c} -s & \cdot & \frac{1}{2} & s \\ \hline \cdot & -\frac{1}{2} & \cdot & \cdot \\ \hline \frac{1}{2} & \cdot & c & c \\ \hline s & \cdot & c & \cdot \end{array}\right) \\
\xi_3=\left(\begin{array}{c|c|c|c} -\frac{1}{2} & \cdot & \cdot & \cdot \\ \hline \cdot & c & \frac{1}{2} & c \\ \hline \cdot & \frac{1}{2} & -s & s \\ \hline \cdot & c & s & \cdot \end{array}\right) &
\xi_4=\left(\begin{array}{c|c|c|c} -\frac{1}{2} & \cdot & \cdot & \cdot \\ \hline \cdot & -\frac{1}{2} & \cdot & \cdot \\ \hline \cdot & \cdot & -\frac{1}{2} & \cdot \\ \hline \cdot & \cdot & \cdot & \frac{3}{2} \end{array}\right)
\end{array}\right\} \tag{33}$$

where
$$s=\sin\frac{\pi}{10}=\frac{\sqrt{5}-1}{4};\quad c=\cos\frac{2\pi}{10}=\frac{\sqrt{5}+1}{4}.$$

In the above representation ξ_4 is taken as diagonal. We can also derive a non-diagonal representation of our ξ's by making use of the following considerations. In virtue of (2) the β_μ along with the $t_{\mu\nu}$ form ten infinitesimal transformations I^{KL} (see Bhabha 1945*b*, § 2, and 1945*c*, p. 244) of the nucleus of an irreducible representation of the Lorentz group in five dimensions with K and L taking values from 1 to 5 (or in Bhabha's notation, from 0 to 4). The division of these ten transformations into two sets of four and six is achieved by merely writing $I^{\mu 5} = \beta_\mu$, i.e. denoting the set I^{15}, I^{25}, I^{35}, I^{45} by $\beta_1, \beta_2, \beta_3, \beta_4$. A completely equivalent way of the subdivision of the I^{KL} would be to take I^{12}, I^{13}, I^{14}, I^{15} in place of the β_μ's. In the notation of our ξ-algebra, this means that in place of $\xi_1, \xi_2, \xi_3, \xi_4$ as fundamental symbols generating the algebra, we might take ξ_1, $-[\xi_1, \xi_2]$, $-[\xi_1, \xi_3]$, $-[\xi_1, \xi_4]$. We now write ξ_1 in the form (30) and make a suitable choice of the p, q, r appearing therein, and thus obtain the matrix for ξ_1. For example a particular choice yields

$$\xi_1 = \left(\begin{array}{c|c|c|c} 2 & \dfrac{i\sqrt{5}}{2} & -\dfrac{i\sqrt{5}}{2} & \dfrac{\sqrt{5}}{2} \\ \hline \dfrac{i\sqrt{5}}{2} & -1 & \frac{1}{4} & \dfrac{i}{2} \\ \hline -\dfrac{i\sqrt{5}}{2} & \frac{1}{4} & -1 & -\dfrac{i}{2} \\ \hline \dfrac{\sqrt{5}}{2} & \dfrac{i}{2} & -\dfrac{i}{2} & \cdot \end{array}\right). \qquad (33')$$

The matrices for ξ_2, ξ_3, ξ_4 are then obtained so as to be in consonance with the fact that the ξ_1, $-[\xi_1, \xi_2]$, $-[\xi_1, \xi_3]$ and $-[\xi_1, \xi_4]$ satisfy the same commutation rules as $\xi_1, \xi_2, \xi_3, \xi_4$. The representation thus obtained is non-diagonal.

By taking the direct product with the Dirac matrices η_μ, we can obtain from (33) and (33′) two equivalent types of representation matrices of order 16 for the β_μ in one of which β_4 is diagonal and not so in the other. By a permutation of rows and columns, we have been able to show that the matrices of the former type for β_k can be brought to the form given by Bhabha (1945*c*, p. 251, (35)) in the case of general spin (with α^0 diagonal). For the case of spin $\frac{3}{2}$ in particular, Bhabha has further given (1945*c*, p. 261, (75), (76)) representation matrices for the β_k expressed in terms of the u and v matrices introduced by Dirac (1936) and Fierz (1939). Writing these matrices in numerical form using the expressions given by Fierz (ibid. Appendix) we can derive the corresponding 16×16 matrices of the representation in which β_4 is not diagonal. It is also possible to show that the second type of matrices which we obtain from (33′) would be equivalent, but for permutation of rows and columns, to the above matrices derived from those given by Bhabha.

Fifth order. Proceeding as in the case of the fourth order representation, and taking ξ_4 diagonal, we deduce that, consistent with the commutation rules, the form of ξ_4 is uniquely determined by taking the eigen-value $-\frac{1}{2}$ as appearing thrice and $\frac{3}{2}$

twice. This is also in consonance with Bhabha's result (1945*c*, p. 259) that the eigenvalues of β_4 in the diagonal form are $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ appearing 4, 6, 6, 4 times respectively. Thus we write

$$2\xi_4 = \operatorname{diag}(-1^3, 3^2),$$

and find that the ξ_k $(k = 1, 2, 3)$ are given by the same expressions as (30) where

$$P_1 = \frac{\sqrt{3}}{2}\begin{pmatrix} p & a \\ q & b \\ p-q & a-b \end{pmatrix}, \quad P_2 = \frac{\sqrt{3}}{2}\begin{pmatrix} -p & -a \\ -q & -b \\ p-q & a-b \end{pmatrix}, \quad P_3 = \frac{\sqrt{3}}{2}\begin{pmatrix} -p & -a \\ q & b \\ q-p & b-a \end{pmatrix}, \tag{34}$$

with a, b, p, q, subject only to the conditions

$$\left.\begin{aligned} a^2+b^2-ab &= p^2+q^2-pq = \tfrac{1}{2}, \\ 2(ap+bq) &= aq+bp. \end{aligned}\right\} \tag{35}$$

We make the particular choice

$$p = q = \frac{1}{\sqrt{2}}, \quad a = -b = \frac{1}{\sqrt{6}},$$

so as to obtain the matrices Hermitian (in fact real and symmetric). This gives

$$\left.\begin{aligned}
2\xi_1 &= \left(\begin{array}{c|c|c|c|c}
1 & 1 & 1 & \sqrt{(\frac{3}{2})} & (1/\sqrt{2}) \\ \hline
1 & 1 & -1 & \sqrt{(\frac{3}{2})} & -(1/\sqrt{2}) \\ \hline
1 & -1 & 1 & \cdot & (\sqrt{2}) \\ \hline
\sqrt{(\frac{3}{2})} & \sqrt{(\frac{3}{2})} & \cdot & \cdot & \cdot \\ \hline
(1/\sqrt{2}) & -(1/\sqrt{2}) & (\sqrt{2}) & \cdot & \cdot
\end{array}\right), \quad
2\xi_2 = \left(\begin{array}{c|c|c|c|c}
1 & 1 & -1 & -\sqrt{(\frac{3}{2})} & -(1/\sqrt{2}) \\ \hline
1 & 1 & 1 & -\sqrt{(\frac{3}{2})} & (1/\sqrt{2}) \\ \hline
-1 & 1 & 1 & \cdot & (\sqrt{2}) \\ \hline
-\sqrt{(\frac{3}{2})} & -\sqrt{(\frac{3}{2})} & \cdot & \cdot & \cdot \\ \hline
-(1/\sqrt{2}) & (1/\sqrt{2}) & (\sqrt{2}) & \cdot & \cdot
\end{array}\right) \\
2\xi_3 &= \left(\begin{array}{c|c|c|c|c}
1 & -1 & 1 & -\sqrt{(\frac{3}{2})} & -(1/\sqrt{2}) \\ \hline
-1 & 1 & 1 & \sqrt{(\frac{3}{2})} & -(1/\sqrt{2}) \\ \hline
1 & 1 & 1 & \cdot & -(\sqrt{2}) \\ \hline
-\sqrt{(\frac{3}{2})} & \sqrt{(\frac{3}{2})} & \cdot & \cdot & \cdot \\ \hline
-(1/\sqrt{2}) & -(1/\sqrt{2}) & -(\sqrt{2}) & \cdot & \cdot
\end{array}\right), \quad
2\xi_4 = \left(\begin{array}{c|c|c|c|c}
-1 & & & & \\ \hline
 & -1 & & & \\ \hline
 & & -1 & & \\ \hline
 & & & 3 & \\ \hline
 & & & & 3
\end{array}\right)
\end{aligned}\right\} \tag{36}$$

Similar remarks apply here, as regards non-diagonal representations, as for the fourth order.

We next consider the spurs of the independent elements of the basis of the algebra A. Since the spurs of all the basis elements of the Dirac algebra D_η except I are zero, it follows from (12) that the only elements of the basis of A which have non-vanishing spurs are those which have non-vanishing spurs among the ξ's. We therefore list

below the spurs of the elements of the basis of the ξ-algebra in the three irreducible representations:

$$\left.\begin{array}{lccc} \text{representation} & \text{1-row} & \text{4-row} & \text{5-row} \\ \text{spur } I & 1 & 4 & 5 \\ \text{spur } \xi_{\mu} & -\frac{1}{2} & 0 & \frac{3}{2} \\ \text{spur } \xi_{\mu\nu} & \frac{1}{4} & 0 & -\frac{3}{4} \\ \text{spur } \xi_{\mu\nu\rho} & -\frac{1}{8} & \frac{1}{2} & -\frac{5}{8} \\ \text{spur } \xi_{\mu\nu\rho\epsilon} & \frac{1}{16} & -\frac{1}{4} & \frac{5}{16} \end{array}\right\}. \tag{37}$$

By means of these, we can write down immediately the spurs of all combinations of the β's by using the algebraic relations (11) and the fact that all elements of D_{η}, except I have zero spur. Thus for example, the spur calculations relating to products of powers of the several β_{μ} given by Booth & Wilson (1940, pp. 498–9) can be carried out with less labour in our case with the aid of the ξ's than in the case of the 10-row representation of the meson algebra. Similarly for spur calculations relating to Bhabha's new theory of the proton, it would not be necessary to use the 16×16 β-matrices, but only the 4×4 ξ-matrices.

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